Lecture 10: Entropy Compression

KEEP TRACK OF COLORING & UNCOLORING

Idea

- Color vertices of a graph randomly one by one
- If conflict, uncolor some to resolve the conflict
- Keep track of what and why is being uncolored.
- If after running for t steps, number of "keep tracks" is strictly less than number of "random one by one" colorings, there must be a good coloring.

4 colors . . 4 . 4 . 4 . 4 . 4 . . .

RECORD . . 3 . 3 . 3 . 3 . .

A *square* in a coloring φ of a graph G is a path $v_1, v_2, \ldots, v_{2k}$, where $\varphi(v_i) = \varphi(v_{i+k})$. In other words, the coloring on the first half is repeated on the second half. Coloring without squares is *non-repetitive*.

Theorem 1 (Thue)**.** *Every path P_n has a non-repetitive coloring using 3 colors.*

1: Find a non-repetitive coloring of P_8.

1 2 3 2 1 3 1 2

Conjecture 2. *Every path P_n has a non-repetitive list coloring using 3 colors per list.*

Theorem 3. *Every path P_n has a non-repetitive list coloring using 4 colors per list.*

Algorithm to find a coloring φ of $P_n = v_1, v_2, \ldots, v_n$ using lists L. It also produces record R, which is a sequence of 0s and 1s.

- pick smallest i, where $\varphi(v_i)$ is not colored
- pick $\varphi(v_i) \in L$ randomly
- add 0 to R
- if $v_{i-2\ell+1}, \ldots, v_i$ is a square then
  - uncolor $v_{i-\ell+1}, \ldots, v_i$
  - add ℓ times 1 to R

4 colors

$R = 0100101100\,0\,1$

$R = \_\_\_\_\_\_ 0\,111\,000\,111$

Note that after each run, φ is non-repetitive.

2: Show that if $\varphi(v_i)$ is colored and it creates a square, there is exactly one square.

. . . 4 . 4 . 4 .

Think of $t > n$. Maybe way bigger.

3: Let R_t and φ_t be R and φ after t iterations of the algorithm, where no coloring of P_n was obtained. Show that from R_t and φ_t, one can reconstruct all choices that were made for $\varphi(v_i)$ during the t steps.

φ_t 1 2 3 φ_{t+1} 2 | $\boxed{3}$

CAN YOU RECONSTRUCT ALL CHOICES OF $\varphi(v_i)$?

0 100 1100 11 0110 1 $\boxed{0}$

0 $\boxed{0}$ 1 1

R_{t-1}

4: Show that there are 4^t possible runs of the algorithm for t steps.

$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \dots 4^t$

5: What are upper bounds on the number of possibilities for φ_t, what is the length of R_t? How does number of 0s and 1s compare in R_t?

$\varphi_t \dots \le (4+1)^n = 5^n$

#0 $\ge$ #1

00101 . .

$|R_t| \le 2t$

6: Show that there is a bijection between R_t and lattice paths from $(0,0)$ to (t,t) that are not going above diagonal. These are counted by the Catalan numbers $\frac{1}{t+1}\binom{2t}{t} \approx \boxed{\frac{4^t}{t^{3/2}\sqrt{\pi}}}$

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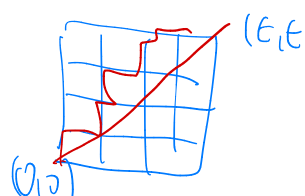


7: Finish the proof by comparing the number of possible records and number of possible runs.

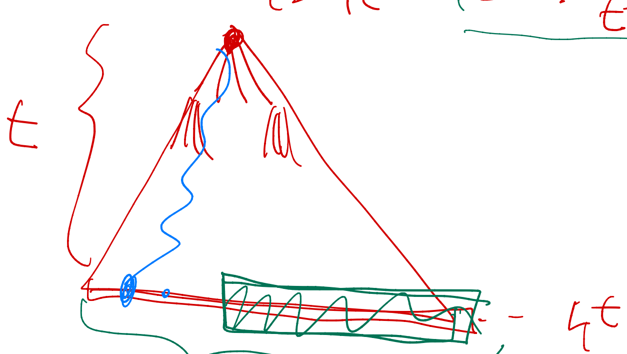
POSSIBLE RUNS . . 4^t

FOR LARGE t

$\vee$

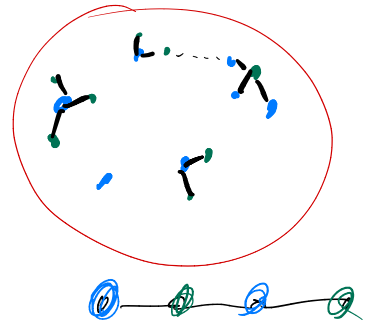
POSSIBLE R_t & φ_t $\quad \boxed{5^n \cdot \frac{4^t}{t^{3/2}\sqrt{\pi}}}$

n IS CONSTANT
ONLY t VARIABLE



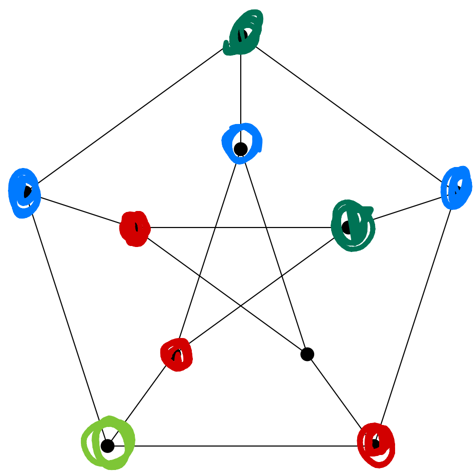
A proper graph coloring φ is a *star coloring* if the union of every two color classes induces a star forest. In other words, vertices of every path on 4 vertices have at least 3 colors.

Related to *acyclic coloring*, where every two color-classes induce a forest.



8: Determine the star chromatic number for a path.

9: Determine the star chromatic number for the Petersen graph.



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Theorem 4. *If G is a graph of maximum degree at most d, then G has a star coloring using at most $\lceil 100d^{3/2} \rceil$ colors.*

Proof. Let $q = \lceil 100d^{3/2} \rceil$ be the number of available colors and $n = |V(G)|$. Order the vertices $v_1, \ldots, v_n$ in an arbitrary (but fixed) order. n^ℓ

Notice that a path $v_i, x_1, x_2, \ldots, x_\ell$, if v_i is known can be described using d^ℓ choices. We call if $d(v_i, x_1, x_2, \ldots, x_\ell)$

We start with φ not assigning any color to any vertex and R being an empty record.

φ ... coloring
R ... record

v_1 v_2 v_3 v_n

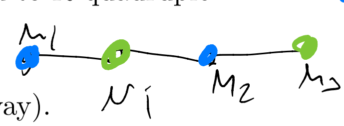
1 step

- let v_i be the uncolored vertex with smallest index
- give v_i a random color from $[q]$ and add to R record $[color]$.
- if there exists an edge uv_i, where $\varphi(u) = \varphi(v_i)$, add to R triple $[Uncolor1, d(uv), \varphi(u)]$, and uncolor both u and v_i.

  v_i u
- if there exists a path $v_iu_1u_2u_3$, where $\varphi(v_i) = \varphi(u_2)$ and $\varphi(u_1) = \varphi(u_3)$, add to R quadruple $[Uncolor2, d(v_iu_1u_2u_3),$ and uncolor all of $v_iu_1u_2u_3$.

  $\varphi(u_1)\varphi(u_3)$
- if there exists a path $u_1v_iu_2u_3$, where $\varphi(v_i) = \varphi(u_3)$ and $\varphi(u_1) = \varphi(u_2)$, add to R quadruple $[Uncolor3, d(v_iu_1)d(v_i, u_2u_3), \varphi(v_i), \varphi(u_1)]$, and uncolor all of $u_1v_iu_2u_3$.

If there are more candidates for uncoloring pick just one (in some deterministic way).

Let the coloring procedure run for t steps. In each step, one vertex is color. i.e. there is a sequence of vertices $v_1 = x_1, x_2, \ldots, x_t$ that get colored by colors $c_1, c_2, \ldots, c_t$ during the second step of the algorithm.

10: How many different possibilities are the for the algorithm to run? What is the probability of one particular run?

1 ITERATION .. q $\quad$ t ITERATIONS q^t $\quad$ $\frac{1}{q^t}$

Let φ_t and R_t be the partial coloring and record after t steps.

11: Show that φ_t and R_t exactly determine the sequence $\boxed{c_1, \ldots, c_t}$. Hint: First determine that it gives the order $v_1 = x_1, x_2, \ldots, x_t$. Show that the colors $c_1, \ldots, c_t$ can be also reconstructed. Notice they are not saved in step 2.

$t \gg n$

$\varphi_1 = [2\ x\ x\ x\ x]$

$\varphi_2 = [2\ 2\ x\ x\ x]$

$t = 2$

$R =$ COLOR COLOR $\quad c_1 = a$, $c_2 = b$

$\varphi_t = \varphi_2[a, b, x\ x\ x]$

R: COLOR (v_1) — φ_R

COLOR (v_2)

UNCOLOR $(v_2 .. d^3)$ COLOR $\varphi(v_2)$

COLOR (v_1)

$c_1 = c$

$c_2 = c$

$x_1 = v_1$

$x_2 = v_2$ COLOR

$x_3 = v_3$

$\varphi_1\ [2\ x\ x\ x\ x]$

$\varphi_2\ [2\ 2\ x\ x\ x]$

UNCOL ... $(v_1 v_2 c)$

IF NO UNCOLOR $\varphi_t(v_1) =$

$\varphi_2 [x\ x\ x\ x]$

$x_1 = v_1$ $\quad$ $\varphi_3 = [2\ x\ x\ x]$

12: What is the number of possibilities for φ_t?

$(q+1)^n$ ← CONSTANT

13: How many times at most *Color* and *Uncolor*? appears in R_t? What is the number the possible sequences of these operations?

COLOR ... t ---> $\frac{3}{2}t$ ENTRIES C & UNC $\quad$ $4^{\frac{3}{2}t}$

UNCOLOR ... $t/2$

R [C U C U2 C U3 C]

14: How many times a particular color is mentioned in R_t? What is the number of choices?

$t/2$ COLORS MENTIONED $\quad$ $q^{t/2}$

15: How many times a particular codes for paths appear in R_t? What is the number of choices?

UNCOLOR 1 ..... d ...... $t/2$ $\quad$ $d^{t/2}$ ...

UN COLOR 2, 3 .. d CHOSEN 3x $t/4$ ..... $(d^3)^{t/4}$ .. $d^{3t/4}$

CHOICES FOR 'PATHS' IN UNCOLOR

16: Combine the choices and derive what is the number of possibilities for R_t and φ_t. These are the failed runs. Calculate the probability that a run is failing after t steps, i.e. divide by the total number of runs. And finish the proof.

$$R_t \,\&\, \varphi_t : \quad d^{3t/4} \cdot q^{t/2} \cdot 4^{3/2 t} \cdot (q+1)^4 \geq \#\text{ FAILD RUNS}$$

(NO COLORING IN t STEPS)

$$P(\text{FAILD RUN}) = \frac{d^{3t/4} \cdot q^{t/2} \cdot 4^{3/2 t} (q+1)^4}{q^t}$$

$$= d^{3t/4} \cdot q^{-t/2}\, 4^{3/2 t} (q+1)^4$$

$$= d^{3t/4} \cdot (100\, d^{3/2})^{-t/2}\, 4^{3/2 t} (q+1)^4$$

$$= 100^{-t/2} \cdot 4^{3/2 t} (q+1)^4 =$$

$$= 100^{-t/2} \cdot (4^{-3})^{-t/2} (q+1)^4$$

$$= \left(\frac{100}{4^3}\right)^{-t/2} (q+1)^4 = \left(\frac{100}{64}\right)^{-t/2} (q+1)^4$$

$$\left(\sqrt{\frac{64}{100}}\right)^t (q+1)^4 \to 0$$

< 1 $\quad \left(\frac{4}{5}\right)^t$

CONSTANT

$\square$